Discrete Mathematics

Planar Graphs

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Planar Graphs

A graph is planar if it can be drawn on paper without edges crossing

Such a drawing is a planar representation

A planar graph may come disguised in a non-planar representation

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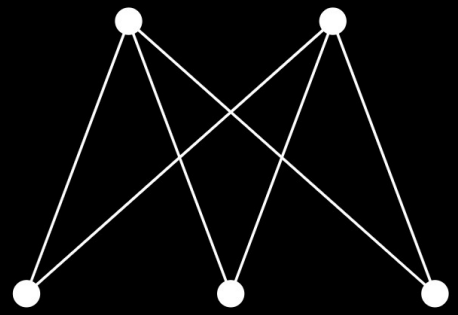
Planar Graphs

Is the graph planar?



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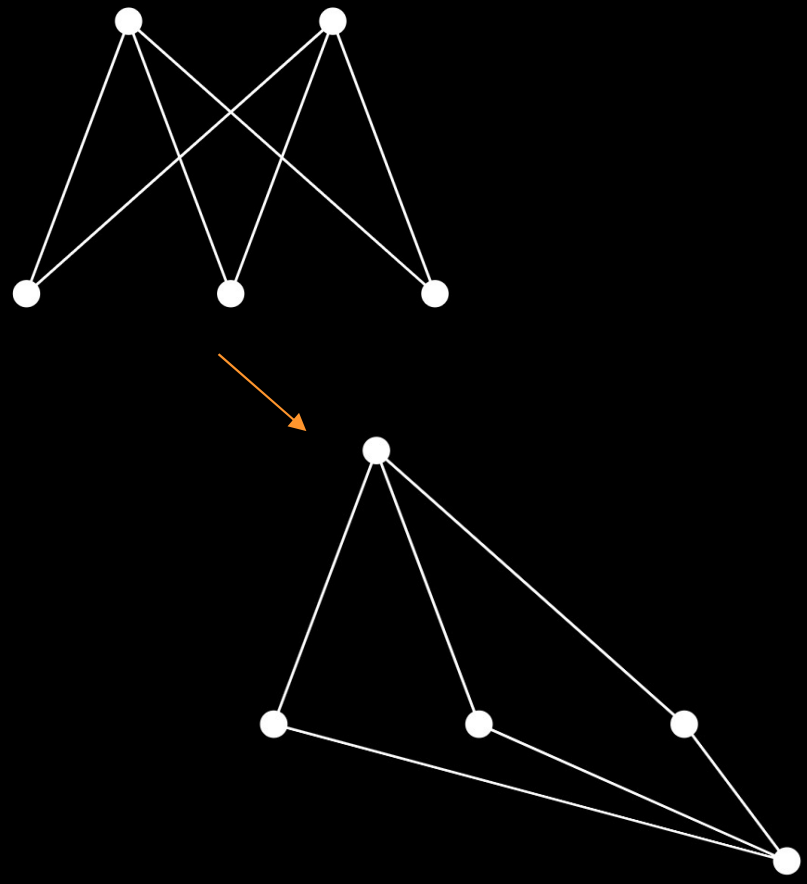
Planar Graphs

Is the graph planar?

Yes!



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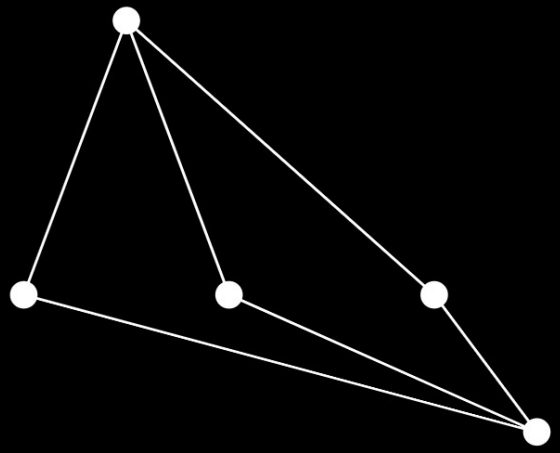
Faces

When a planar graph is drawn in its planar representation, it divides the plane in to faces

How many faces does this graph have?



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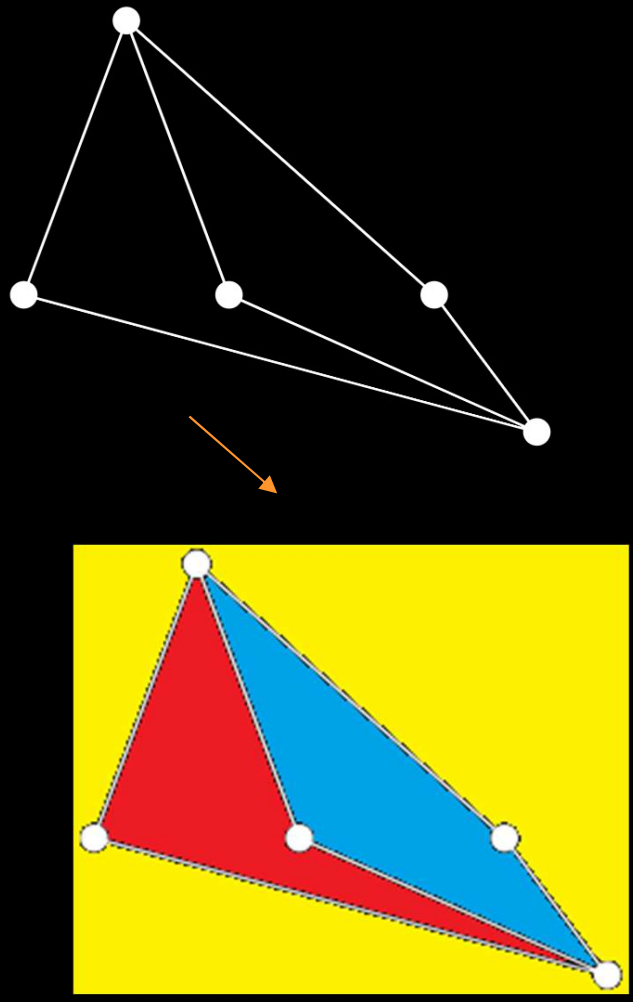
Faces

When a planar graph is drawn in its planar representation, it divides the plane in to faces

How many faces does this graph have?

3 faces



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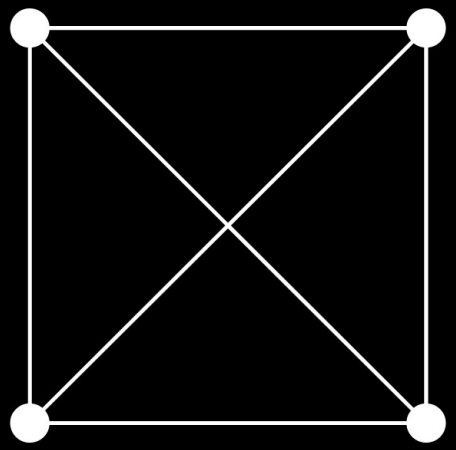
Faces

How many faces does this graph have?



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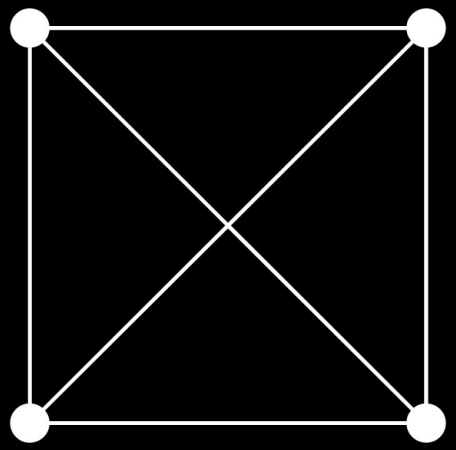
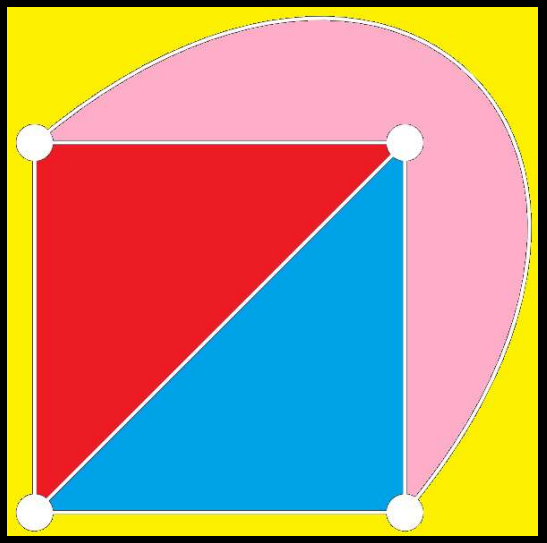
Faces

How many faces does this graph have?

4





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Euler's Formula

For a connected planar graph with v vertices, e edges, and f faces

$$v - e + f = 2$$

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Euler's Formula

Sketch of proof:

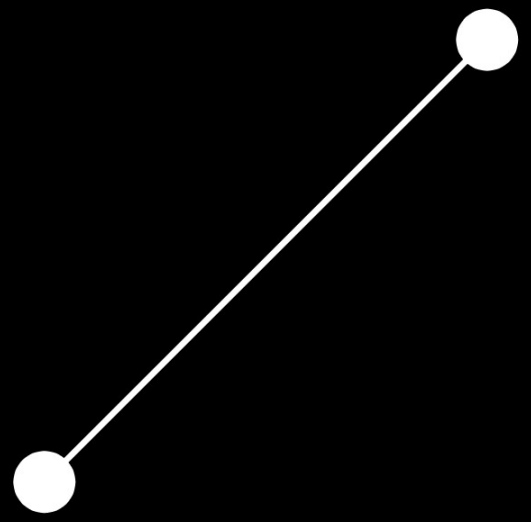
First, start with P_1.

$v = 2$

$e = 1$

$f = 1$

$v - e + f = 2 - 1 + 1 = 2$



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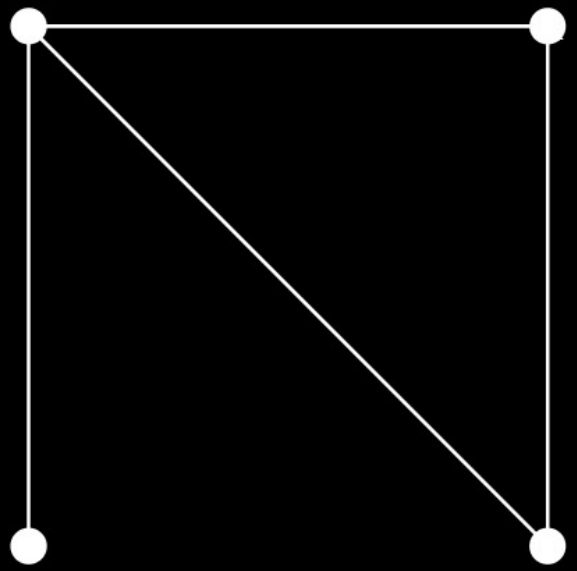
Euler's Formula

Given any planar graph, how can we "add an edge?"



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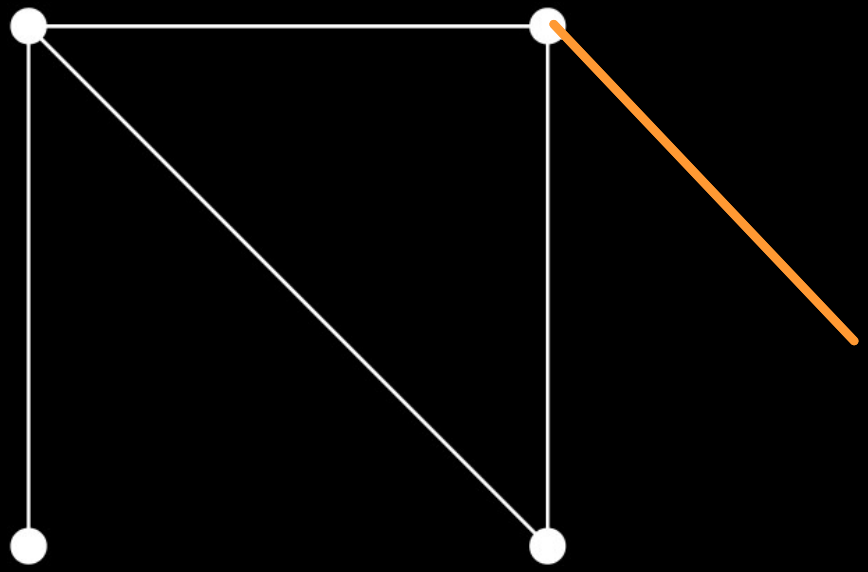
Euler's Formula

Given any planar graph, how can we "add an edge?"

Two ways…

Way #1 → Add a "finger"



v increases by 1
e increases by 1
f remains the same

$v - e + f$ remains the same (2)

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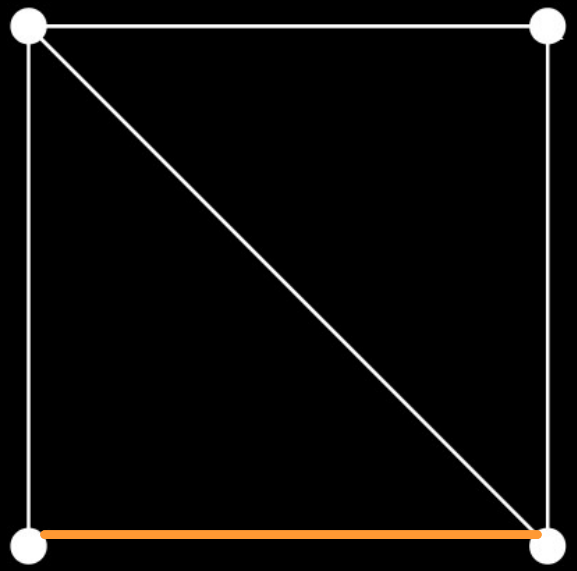
Euler's Formula

Given any planar graph, how can we "add an edge?"

Two ways…

Way #2 → close a loop



v stays the same
e increases by 1
f increases by 1

$v - e + f$ remains the same (2)

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Euler's Formula

For a connected planar graph with v vertices, e edges, and f faces

$$v - e + f = 2$$

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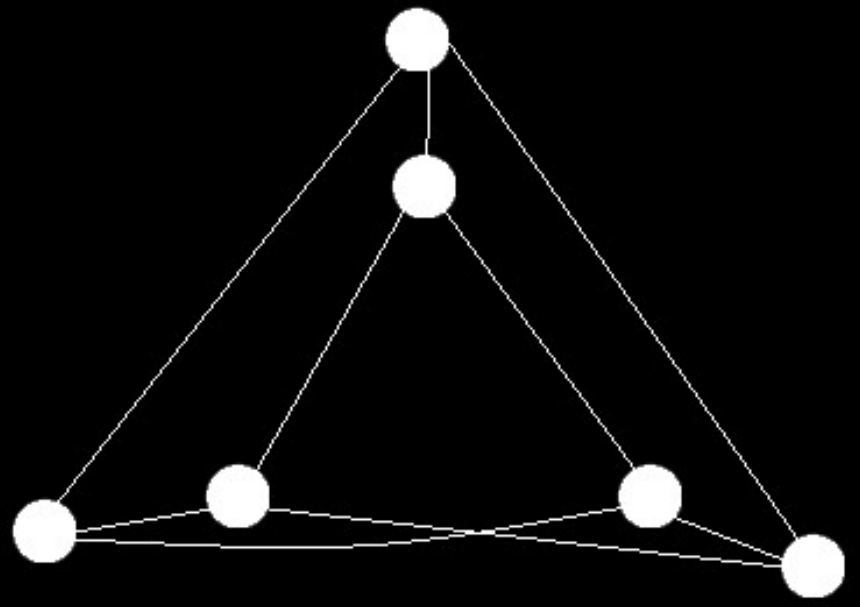
Girth

In any graph, the girth of the graph is the size of its smallest cycle.

What is the girth of this graph?



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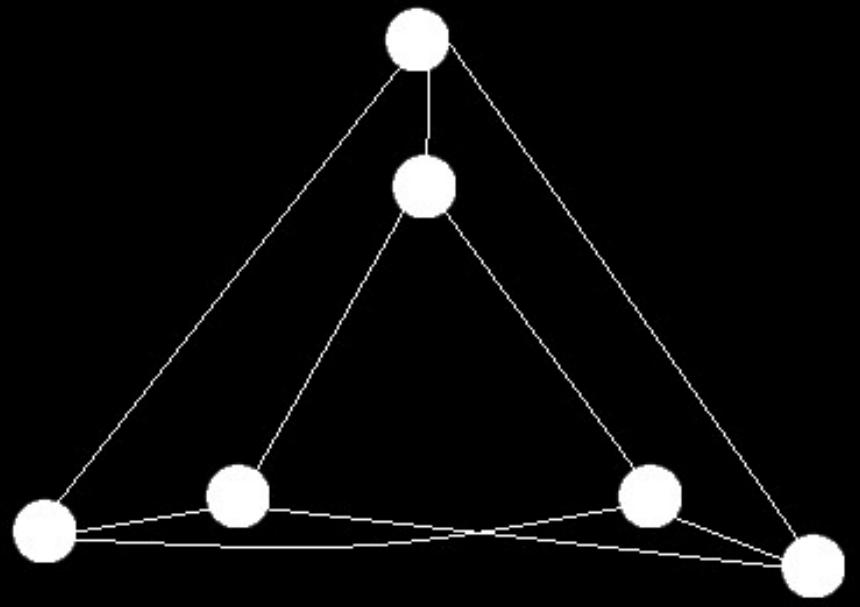
Girth

In any graph, the girth of the graph is the size of its smallest cycle.

What is the girth of this graph?

4



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Theorem

A connected planar graph with f faces and of girth g has:

$$gf \leq 2e$$

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$gf \leq 2e$

Sketch of proof: Let B be the number of boundaries between faces

$$B = 2e$$

$$gf \leq B$$

So $gf \leq 2e$.

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Recap

(Euler's Formula)

For a connected planar graph with v vertices, e edges, and f faces

$$v - e + f = 2$$

A connected planar graph with f faces and of girth g has:

$$g f \leq 2e$$

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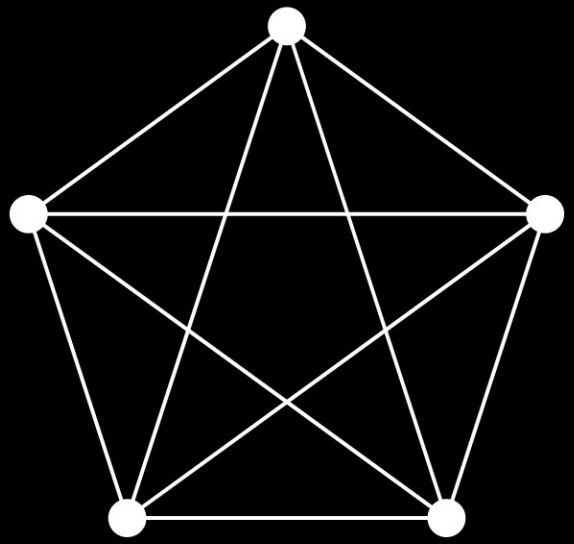
Example

Is K_5 planar?



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Example

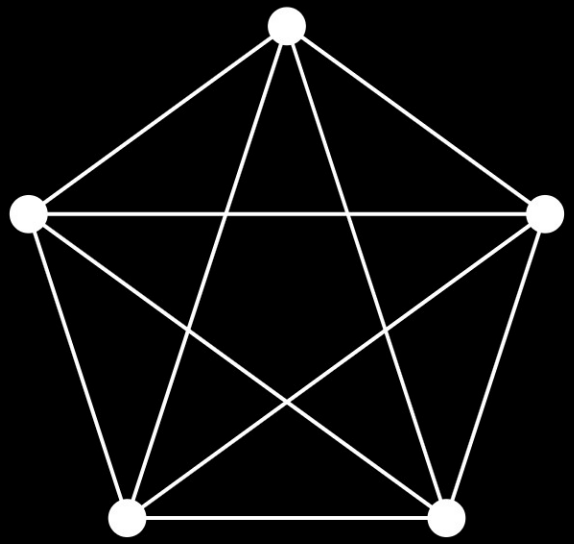
Is K_5 planar?

If it was…

$v = 5, e = 10, g = 3$

$v - e + f = 2$ (Euler)

So $f = 7$



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Example

Is K_5 planar?

If it was…

$v = 5$, $e = 10$, $g = 3$

$v - e + f = 2$ (Euler)

So $f = 7$

$g f \leq 2e$

$3\ (7) \leq 2\ (10)$

$21 \leq 20$ Contradiction!

So K_5 is not planar

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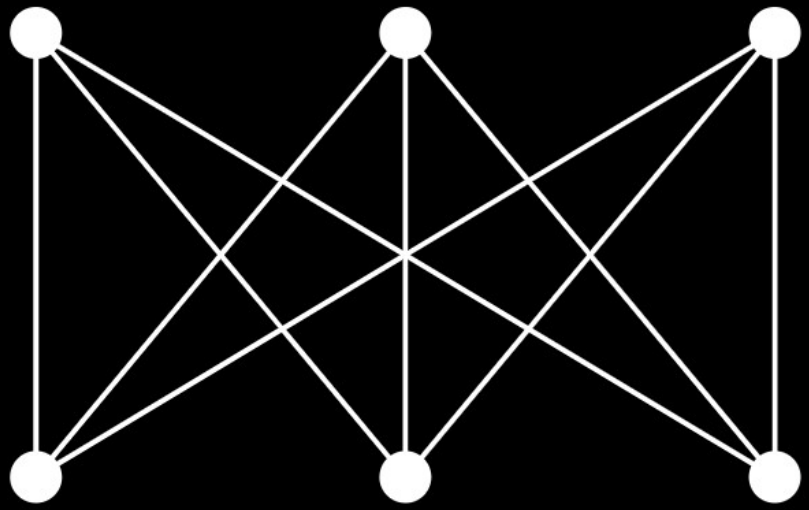
Example

Is $K_{3,3}$ planar?



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Example

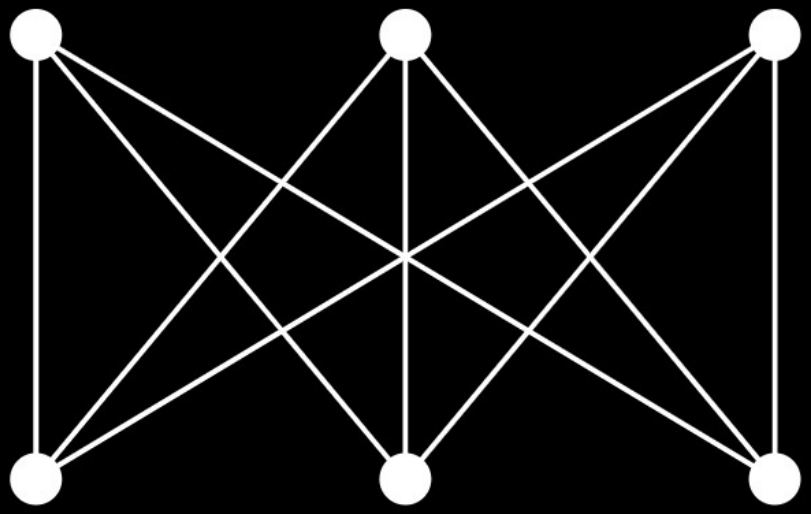
Is $K_{3,3}$ planar?

If it was…

$v = 6, e = 9, g = 4$

$v - e + f = 2$ (Euler)

So $f = 5$



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Example

Is $K_{3,3}$ planar?

If it was…

$v = 6$, $e = 9$, $g = 4$

$v - e + f = 2$ (Euler)

So $f = 5$

$g f \leq 2e$

$4\ (5) \leq 2\ (9)$

$20 \leq 18$ Contradiction!

So $K_{3,3}$ is not planar

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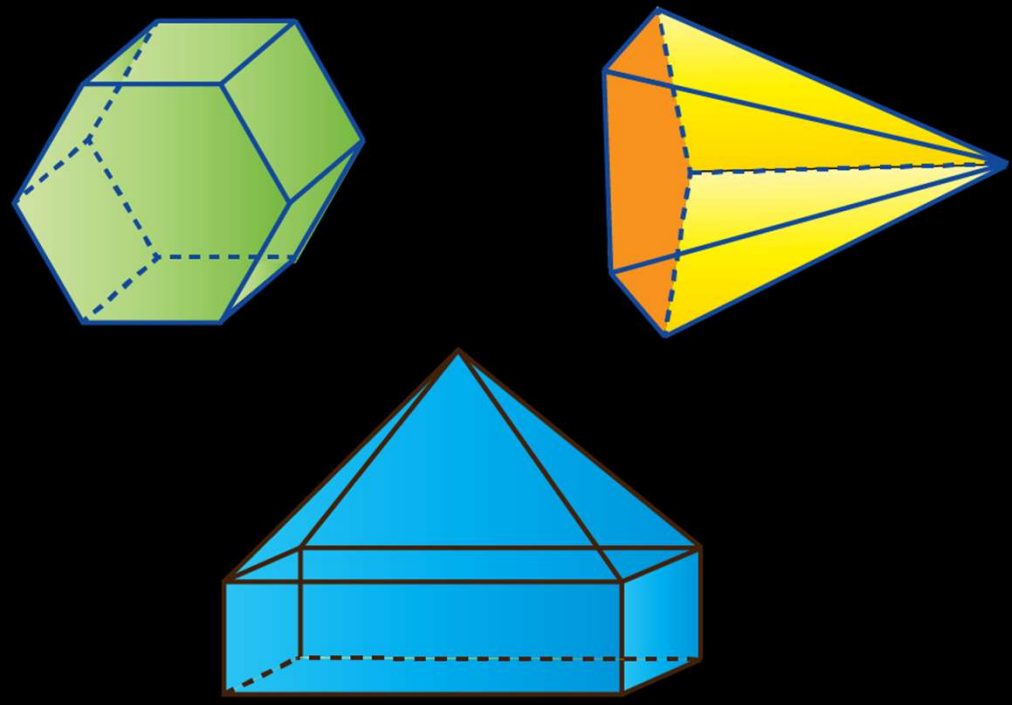
Polyhedra

A polyhedron is a three-dimensional geometric object constructed of flat polygonal faces connected by vertices and edges.

(Namely, a particular kind of graph)



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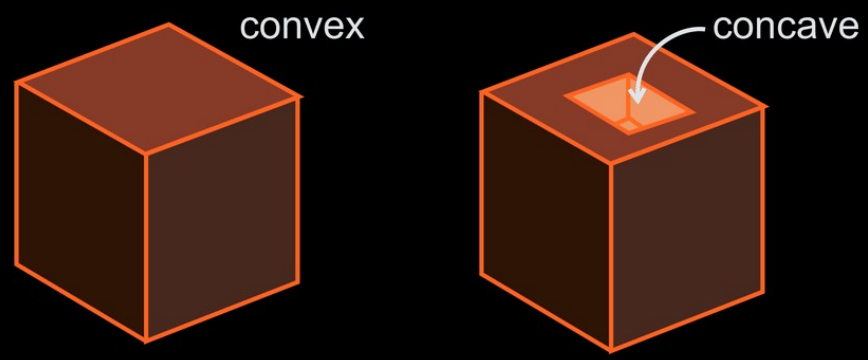
Convex and Concave

A polyhedron is convex if a line segment connecting two points within the polyhedron lies entirely within the polyhedron.

(Otherwise it is concave)



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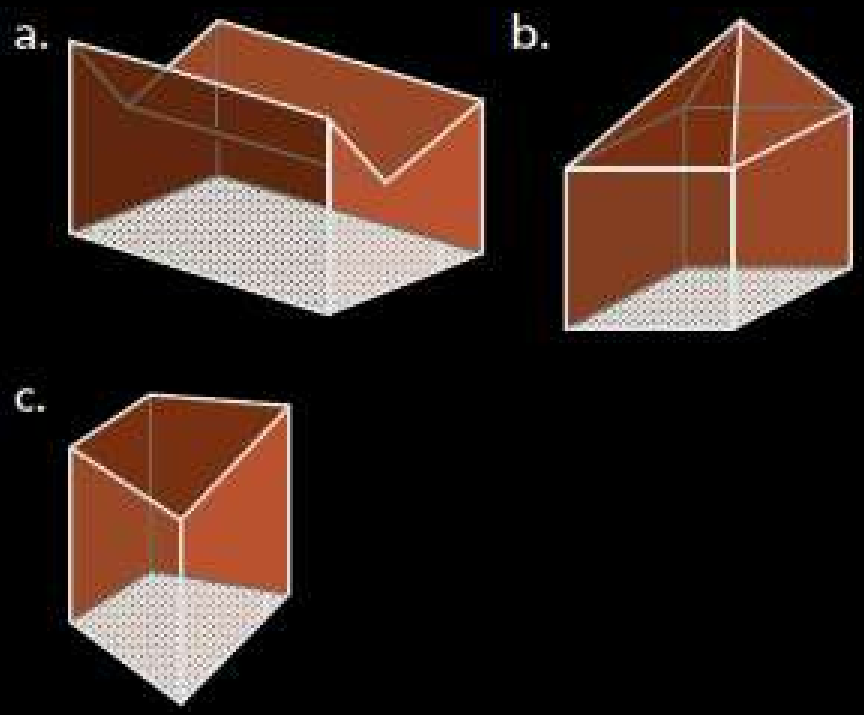
Convex and Concave

Are the polyhedral convex or concave?



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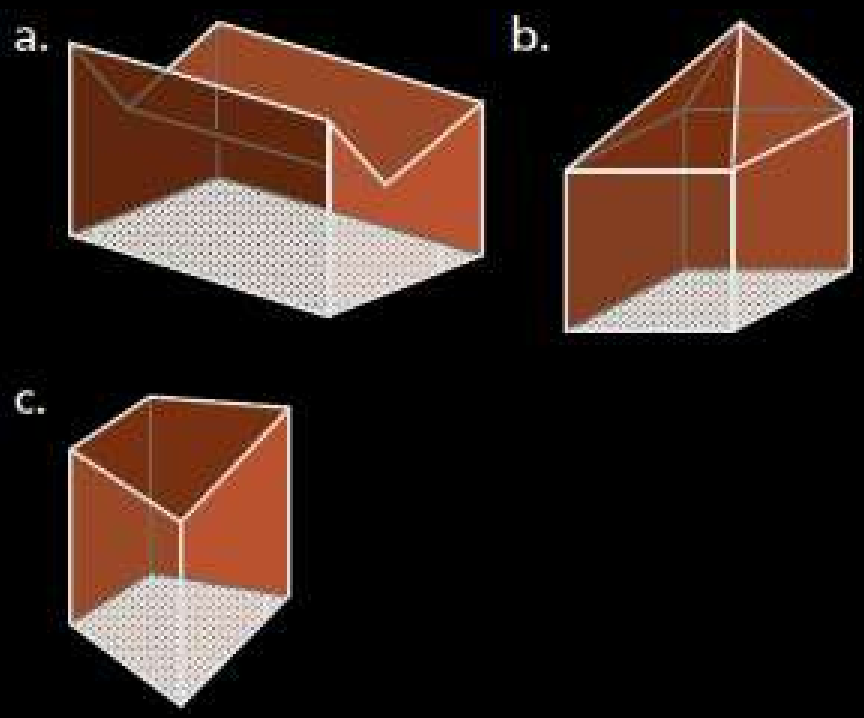
Convex and Concave

Are the polyhedral convex or concave?

a) Concave
b) Convex
c) Convex



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Theorem

Every convex polyhedron is a planar graph.

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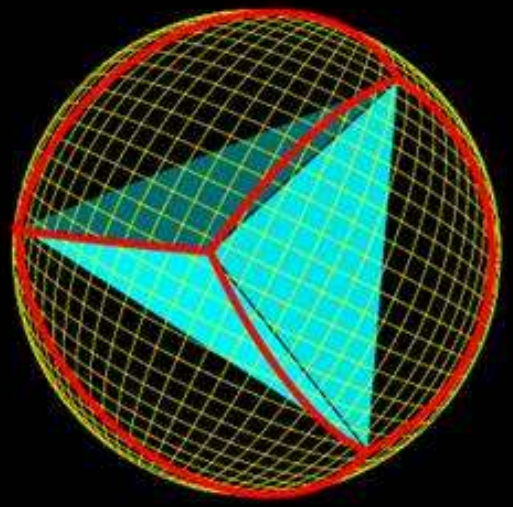
Convex Polyhedra are Planar

Sketch of proof:

Embed polyhedron in sphere, place “bulb” at center of polyhedron

Trace shadow cast upon sphere, then cut open sphere to get plane



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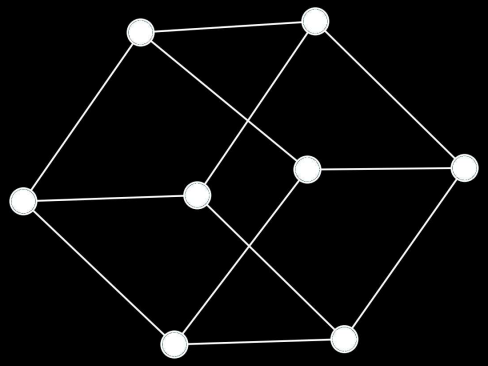
Example

Give a planar representation of the graph represented by a cube.



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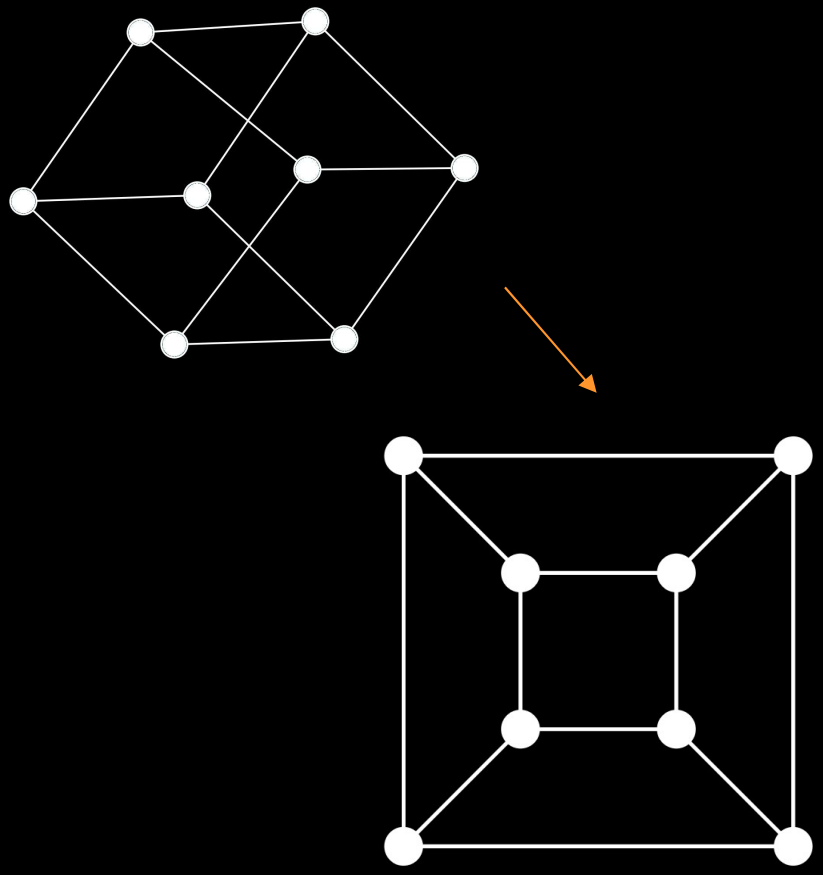

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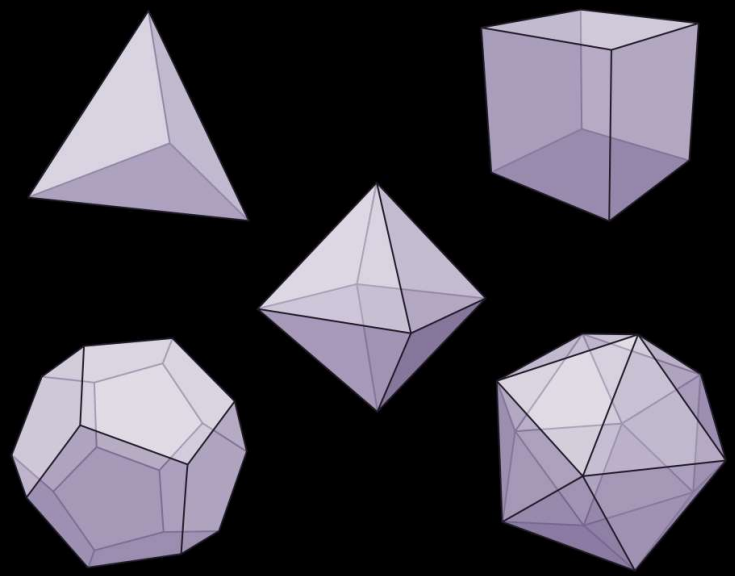
Platonic Solids

A regular polyhedron or Platonic solid is a polyhedron where:

- Each face is a regular polygon
- Each face is identical
- Each vertex joins the same number of faces



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Theorem

There are exactly five platonic solids.

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Theorem

There are exactly five platonic solids.

Sketch of proof: use Euler's formula and you will find five possible solutions. (Refer to book for complete proof).

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