Discrete Mathematics **Homework #14** **Name: ________________**

Question 1 Find a closed formula for the recurrence relation using telescoping.

$a_n = a_{n-1} + 2^n$
$a_0 = 5$

Question 2 Find a closed formula for the recurrence relation using telescoping. Hint: divide both sides of the recurrence relation by 2^n.

$b_n = 2b_{n-1} + 2^n$
$b_0 = 5$

Question 3 Use telescoping to express the given sequence in terms of the harmonic numbers, where the nth harmonic number is given by $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Hint: begin by dividing both sides of the recurrence relation by $(n + 1)$.

$c_n = \left(1 + \frac{1}{n}\right) c_{n-1} + 2$
$c_0 = 0$

Question 1 Solution

$a_n = a_{n-1} + 2^n$
$a_0 = 5$

Let's rewrite the recurrence relation in terms of $a_n - a_{n-1} = 2^n$. Now we can add terms of the sequence for indices $1, \dots, n$ and telescope:

$$a_1 - a_0 + a_2 - a_1 + a_3 - a_2 + \cdots + a_n - a_{n-1} = 2^1 + 2^2 + 2^3 + \cdots + 2^n$$

Cancelling terms, we get:

$$a_n - a_0 = 2^1 + 2^2 + 2^3 + \cdots + 2^n$$
$$a_n - 5 = 2^1 + 2^2 + 2^3 + \cdots + 2^n$$

Let $S = 2^1 + 2^2 + 2^3 + \cdots + 2^n$.

$$a_n = 5 + S$$

We now just need to figure out what S is and then we will have a closed formula for a_n. We will use a clever trick to accomplish this. Let's think of $S = 2S - S$.

Then,

$$S = 2(2^1 + 2^2 + 2^3 + \cdots + 2^n) - (2^1 + 2^2 + 2^3 + \cdots + 2^n)$$
$$S = 2^2 + 2^3 + 2^4 + \cdots + 2^{n+1} - 2^1 - 2^2 - 2^3 - \cdots - 2^n$$

Again, by telescoping, we get:

$$S = 2^{n+1} - 2$$

So our overall closed formula for $a_n = 3 + 2^{n+1}$.

Question 2 Solution

$b_n = 2b_{n-1} + 2^n$
$b_0 = 5$

If we try to rewrite the recurrence relation as $b_n - 2b_{n-1} = 2^n$ we cannot use the telescoping method, as adding these terms from $1,2,\ldots,n$ will not have terms that nicely cancel out.

Instead, we can divide both sides of the recurrence relation by 2^n, giving

$$\frac{b_n}{2^n} - \frac{2b_{n-1}}{2^n} = 1$$
$$\frac{b_n}{2^n} - \frac{b_{n-1}}{2^{n-1}} = 1$$

These terms will telescope nicely.

$$\frac{b_1}{2^1} - \frac{b_0}{2^0} + \frac{b_2}{2^2} - \frac{b_1}{2^1} + \frac{b_3}{2^3} - \frac{b_2}{2^2} + \cdots + \frac{b^n}{2^n} - \frac{b_{n-1}}{2^{n-1}} = 1 + 1 + 1 + \cdots + 1$$

Canceling terms on the left and summing on the right, we find our closed formula.

$$\frac{b^n}{2^n} - \frac{b_0}{2^0} = n$$
$$\frac{b^n}{2^n} - 5 = n$$
$$b_n = (n+5)2^n$$

Question 3 Solution

$c_n = \left(1 + \frac{1}{n}\right) c_{n-1} + 2$
$c_0 = 0$

Let's begin by dividing both sides of the recurrence relation by $n + 1$.

$$\frac{c_n}{n+1} = \frac{\left(1 + \frac{1}{n}\right) c_{n-1}}{n+1} + \frac{2}{n+1}$$
$$\frac{c_n}{n+1} = \frac{\left(1 + \frac{1}{n}\right) c_{n-1}}{n+1} \times \frac{n}{n} + \frac{2}{n+1}$$
$$\frac{c_n}{n+1} = \frac{(n+1)c_{n-1}}{n(n+1)} + \frac{2}{n+1}$$
$$\frac{c_n}{n+1} = \frac{c_{n-1}}{n} + \frac{2}{n+1}$$
$$\frac{c_n}{n+1} - \frac{c_{n-1}}{n} = \frac{2}{n+1}$$

Now we have a representation that will telescope nicely.

$$\frac{c_1}{2} - \frac{c_0}{1} + \frac{c_2}{3} - \frac{c_1}{2} + \frac{c_3}{4} - \frac{c_2}{3} + \cdots + \frac{c_n}{n+1} - \frac{c_{n-1}}{n} = \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \cdots + \frac{2}{n+1}$$

Consolidating terms, cancelling, and rearranging:

$$-c_0 + \frac{c_n}{n+1} = 2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}\right)$$
$$c_n = 2(n+1)\left(-1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}\right)$$
$$c_n = 2(n+1)(H_{n+1} - 1)$$